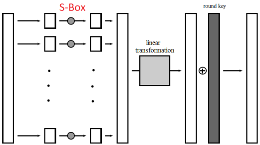
On Some Properties of Quadratic APN Functions of a Special Form

Irene Villa

University of Bergen (Norway)

Cryptography

Cryptography	
> Block ciphers	

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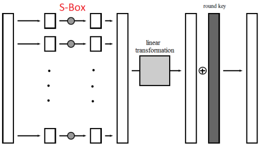
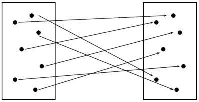


≫ S-Boxes



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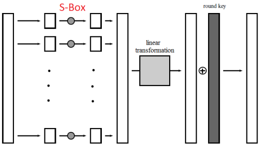
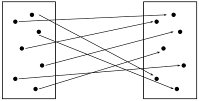
$>$ Block ciphers	
$\gg$ S-Boxes	
$\ggg$ APN functions	optimal resistance against differential attack

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$$Tr_n(x) = x + x^2 + x^4 + \cdots + x^{2^{n-1}}$$

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$F_1, F_2 : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ are CCZ-equivalent ($F_1 \overset{\text{CCZ}}{\sim} F_2$) if
$\mathcal{L}(\Gamma_{F_1}) = \Gamma_{F_2}$, with $\mathcal{L}$ affine permutation of $\mathbb{F}_{2^n}^2$ and
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$$\boxed{F(x) = L_1(x^3) + L_2(x^9)}$$

L_1, L_2 linear functions over $\mathbb{F}_{2^n}$

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(Budaghyan, Carlet and Leander, 2009)

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• $x^3 + a^{-1}Tr_3(a^6x^{18} + a^{12}x^{36})$ is APN for any $a \neq 0$ and $3|n$;
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$n = 8$, $x^9 + Tr_n(x^3)$ is APN
(CCZ-ineq. to power functions and to $x^3 + Tr_n(x^9)$)

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- Last function x^{57} [23] is of algebraic degree 4.

Necessary Conditions

Lemma (1)

For n even, $k = (2^n - 1)/3$ and $\alpha \in \mathbb{F}_{2^n}^*$ primitive element if $F(x) = L_1(x^3) + L_2(x^9)$ is APN then $F(\alpha^j) \neq 0$ for $j = 0, \ldots, k-1$

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For n multiple of 6, if $L_1(x^3) + L_2(x^9)$ is APN then for any $a, \beta \neq 0$ with $Tr_3(\beta) = 0$ $L_1(a^3\beta) \neq 0$

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Proposition

If $L_1(x^3) + L_2(x^9)$ is APN, then the linear function $L_3(x) = L_1(x^2 + x) + L_2(x^8 + x)$ is a 2-to-1 map satisfying $L_3(x) = 0$ if and only if $x = 0, 1$

Necessary and Sufficient Conditions

Lemma (3)

$L_1(x^3) + L_2(x^9)$ is APN if and only if

- *for any $a \neq 0$ and $x \neq 0, 1$*
  $L_1(a^2(x^2 + x)) + L_2(a^9(x^8 + x)) \neq 0$

or equivalently

- *for any $a, y \neq 0$ with $Tr_n(y) = 0$*
  $L_1(a^3 y) + L_2(a^9(y^4 + y^2 + y)) \neq 0$

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Lemma (4)

$L_1(x^3) + L_2(x^9)$ is APN if and only if for any $a \neq 0$ there exists one and only one $\lambda \neq 0$ such that $Tr_n(\lambda L_1(ax^2 + a^2 x) + \lambda L_2(ax^8 + a^8 x)) \equiv 0$

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Lemma (5)

$L_1(x^3) + L_2(x^9)$ is APN if and only if for any $a, y \neq 0$ with $Tr_n(y) = 0$, if it exists $t \in \mathbb{F}_{2^n}$ satisfying $Tr_n(t) = 0$ and $L_1(a^3 y) = L_2(a^9 y^3 t)$ then $L_2(a^9(y^4 + ty^3 + y^2 + y)) \neq 0$

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Corollary

If for any $a, y \neq 0$ $Tr_n(y) = 0$ the equation $L_1(a^3y) + L_2(a^9y^3t) = 0$ is satisfied only for t with $Tr_n(t) = 1$, then $L_1(x^3) + L_2(x^9)$ is APN

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Using Lemma (5) (computational results done with MAGMA)

- $\Rightarrow$ $x^9 + Tr_n(x^3)$ is APN only for $n = 4, 5, 8$ (checked until n=200);
- $\Rightarrow$ list of APN of the form $x^9 + L(x^3)$ (representatives for CCZ-equivalence relation) for $n = 4, \ldots, 10$

On $x^9 + L(x^3)$

CCZ-equivalent classes, with $\alpha \in \mathbb{F}_{2^n}^*$ primitive element

n	$\sharp$	Representative for $L(x)$
4	1	0
5	2	$0, \quad Tr_n(x)$
6	2	$\alpha^{44}x + \alpha x^2, \quad \alpha^{23}x + x^4$
7	1	0
8	8	$0, \quad Tr_n(x), \quad x^2 + x^{16},$ $x^8 + x^{128}, \quad x^4 + \alpha^{85}x^8 + x^{16},$ $\alpha^{60}x + \alpha^{200}x^2 + \alpha^{242}x^4 + \alpha^{190}x^8 + \alpha x^{16},$ $\alpha^{228}x^{64} + \alpha^{107}x^{32} + \alpha^{80}x^8 + \alpha^{137}x^2 + \alpha^{189}x,$ $\alpha^{25}x^{128} + \alpha^{194}x^4 + \alpha^{146}x^2$
9	0	-
10	2	$0, \quad \alpha^{1021}x + \alpha^{1022}x^2 + \alpha x^4$

Fact

If $3 \nmid n$ then $L(x) = 0$ generates the APN function x^9.

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Proposition

If n is even then for any $a \neq 0$ not a cube

$$L(x) = ax^4 + a^{-1}x^2 + a^{-2}x$$

generates an APN function $x^9 + L(x^3)$ linear equivalent to x^3.

Comparison with Edel-Pott list ($n = 6$ and $n = 8$) :

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[$n = 6$]

1. $L(x) = \alpha^{44}x + \alpha x^2$,
   $$x^9 + L(x^3) \overset{CCZ}{\sim} \text{no. } 2\ (x^3 + \alpha^{-1} Tr_n(\alpha^3 x^9)),$$
2. $L(x) = \alpha^{23}x + x^4$,
   $$x^9 + L(x^3) \overset{CCZ}{\sim} \text{no. } 1\ (x^3).$$

$[n = 8]$

1. $L(x) = 0$, no. 2 (x^9),
2. $L(x) = Tr_n(x)$, no. 4 $(x^9 + Tr_n(x^3))$,
3. $L(x) = x^2 + x^{16}$,
   $x^9 + L(x^3) \overset{CCZ}{\sim}$ no. 3 $(x^3 + Tr_n(x^9))$,
4. $L(x) = x^8 + x^{128}$,
   $x^9 + L(x^3) \overset{CCZ}{\sim}$ no. 1 (x^3) ,
5. $L(x) = x^4 + \alpha^{85}x^8 + x^{16}$,
   $x^9 + L(x^3) \overset{CCZ}{\sim}$ no. 6,
6. $L(x) = \alpha^{60}x + \alpha^{200}x^2 + \alpha^{242}x^4 + \alpha^{190}x^8 + \alpha x^{16}$,
   $x^9 + L(x^3) \overset{CCZ}{\sim}$ no. 9,
7. $L(x) = \alpha^{228}x^{64} + \alpha^{107}x^{32} + \alpha^{80}x^8 + \alpha^{137}x^2 + \alpha^{189}x$,
   $x^9 + L(x^3) \overset{CCZ}{\sim}$ no. 5,
8. $L(x) = \alpha^{25}x^{128} + \alpha^{194}x^4 + \alpha^{146}x^2$,
   $x^9 + L(x^3) \overset{CCZ}{\sim}$ no. 19.

Thank you for your attention